*JBE-003-1161002*

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Seat No. ________

## M. Sc. (Sem. I) (CBCS) Examination

**December – 2019**

## Mathematics : Paper - CMT-1002

*(Real Analysis)*

**Faculty Code : 003**

**Subject Code : 1161002**

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

**Instructions :** (1) All questions are compulsory.

(2) Each question carries 14 marks.

1 Answer any seven questions : **7 × 2 = 14**

(i) Define Countable set and give an example of a countable set.

(ii) Define Boolean algebra on a non-empty set $X$.

(iii) Define Borel field and Borel Set.

(iv) Define Outer measure and give an example of an infinite subset of $\mathbb{R}$ whose outer measure is zero.

(v) Give an example of a subset of nowhere dense set.

(vi) Prove or disprove, $\mathbb{R}$ is a measurable set.

(vii) Write down outer measure of following.

sets: Q, [2,5] and (–3,5).

(viii) Is Cantor set measurable? Justify.

(ix) Define almost everywhere property.

(x) Define convergence in sense of measure.

2 Answer any two questions : **2 × 7 = 14**

(a) Let $X$ be a non-empty set and $a$ be a Boolean algebra on $X$. Let $< A_i > \subseteq a$ be any sequence in $a$. Prove that there is a sequence $< B_i >$ in $a$ such that each $B_i$'s are mutually disjoint, $Bi \subseteq Ai$, $\forall\ i \in N$ and

$\bigcup_{i=1}^{n} B_1 = \bigcup_{i=1}^{n} A_i$, for each $n \in N$.

(b) Give an example of a Boolean algebra on N, which is not a $\sigma$-algebra on N. Justify your answer.

(c) Let $F, E \in m$, where $m$ is the collection of all measurable sets. Prove that $F \cup E \in m$.

(d) Prove that the outer measure is translate invariant (i.e. $m^*(A) = m^*(A + y)$, $\forall\ y \in \mathbb{R}$).

3 Answer any one question : **1 × 14 = 14**

(a) Construct a non-measurable subset of [0, 1].

(b) Let $f$ be a bounded function on a measurable set E and $m\,E < \infty$. Prove that $\inf_{\psi \geq f} \int_E \psi \quad \sup_{\phi \leq f} \int_E \phi$,

for all simple functions $\phi$ and $\psi$ on E if and only if $f$ is a measurable function.

(c) State and Prove Vitali's Lemma.

4 Answer any two questions **2 × 7 = 14**

(a) Let $1 \leq p < \infty$. If $f,g \in L^p[0 , 1]$, the prove that $f + g \in L^p[0, 1]$ and $\| f+g \|_p \leq \| f \|_p + \| g \|_p$, where

$$\| f \|_p = \left[ \int_0^1 \ |f|^p \right]^{1/p}.$$

(b) Let $f$ be a bounded measurable function on [a, b] and $F(x) = \int_a^x f(t)dt + F(a), \forall\ x \in [a, b]$. Prove that $F'(X) = f(X)$ almost everywhere on [a, b].

(c) Let $f : [0, 1] \to \mathbb{R}$ and $f(0) = 0$, $f(x) = x^2 \sin(1/x^2)$, $\forall \times E$ (0 , 1]. Prove that f is not a function of bounded variation on [0, 1].

5 Answer any two questions : **$2 \times 7 = 14$**

(a) State and prove Bounded Convergence Theorem.

(b) State and prove Fatou's Lemma.

(c) Let $\{f_n\}$ be a sequence of non-negative measurable functions such that $f_n \leq f_{n+1}$, $\forall\ n \in \mathbb{N}$. Suppose $f_n(X) \to f(X)$, $\forall x \in E$. Prove that $\int_E f = \lim_n \int_E f_n$.

(d) Let $\{f_n\}$ be a sequence of non-negative measurable functions such that fn $f_n \leq f$, $\forall n \in \mathbb{N}$, where f is also a non-negative measurable function. Suppose $f_n(x) \to f(x), \forall x \in E$. Prove that $\int_E f = \lim_n \int_E f_n$.

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